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Counting Kneading Sequences

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1. Introduction

Iterations of continuous maps of an interval $I = [-\beta, \beta]$ into itself serve as the simplest example of models for dynamical systems. These models present an interesting mathematical structure going far beyond the simple equilibrium solutions one might expect. A special case of these kinds of continuous maps are unimodal functions. We say f is *unimodal* if there is a unique *turning point* $x_0 \in I$ such that f is decreasing in $[-\beta, x_0]$ and increasing in $[x_0, \beta]$. A very interesting technique, namely the kneading sequences, has been advocated by Milnor and Thurston in 1978 to study and classify the orbit of unimodal functions. Intuitively, a kneading sequence of an unimodal map is an infinite sequence with symbols in $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$, where L at position i means $f^i(x_0)$ is on the Left of the turning point x_0; R means $f^i(x_0)$ is on the Right of x_0 and C means $f^i(x_0)$ is exactly x_0.

Although the fundamental idea of kneading sequences is simple, it is particularly useful when we are seeking combinatorial classifications of unimodal functions. In this Thesis, we investigate a special family of unimodal functions: the quadratic polynomials $f_c = x^2 + c$ where $c \in [-2, 1/4]$ using the kneading sequence technique. This family is nice because: The kneading sequences all lie in a particular range and because there is a unique function, i.e. a unique c in the family for every periodic kneading sequence. More importantly, the result proved by using the quadratic polynomial family can be extended to any full family of unimodal functions.

The Thesis begins with some background of kneading sequence and one of its modification, called orientation kneading sequence. Results in Section 2 are mostly quoted without giving the detailed verifications. However, the motivations behind these results and their applications to the later sections are introduced. There are three new results presented in this thesis:

- **(A)** (Proposition 3.2 & Theorem 3.4) A new proof of known formulas that count the number of real parameters c such that the quadratic polynomials f_c have a periodic critical point with period n or period exactly n.
- **(B)** (Theorem 4.1) From a kneading sequence, one can extract a description of the itinerary of the critical orbit with respect to a fixed point. This description is called a composition. In the complex plane, every composition has some quadratic function correspond to it. But this is not the case if we restrict to real polynomials. I proved the conditions for a composition to be associated to a real unimodal map.

(**C**) (Proposition 4.3) Each valid composition may arise from several different kneading sequences, so it serves to classify interval maps with similar dynamical behaviors. I gave a partial answer to the number of kneading sequences corresponding to each valid composition.

1.1. **Acknowledgements.** This Thesis is based in joint work with Vorrapan Chandee and Rodrigo Pérez; see [CQP].

2. Kneading sequences

2.1. **Definitions.** We refer the reader to [CE] and [MT] for proofs of the kneading sequence results stated below.

Let $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$ denote the set of infinite sequences in the symbols L, C and R. The *shift* map $\sigma : \{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^* \longrightarrow \{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$ is defined by removing the first symbol of a sequence: $\sigma(u_1u_2\ldots) = u_2u_3\ldots$ We define a special order on $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$ in the following manner: Let $A = a_1a_2\ldots$ and $B = b_1b_2\ldots$ be 2 sequences and j the lowest index such that $a_j \neq b_j$. If ℓ represents the number of L symbols among $a_1, a_2, \ldots, a_{j-1}$, then $A \prec B$ if and only if $a_j < b_j$ according to the rule

$$\begin{array}{c} \mathsf{L} < \mathsf{C} < \mathsf{R} \text{ when } \ell \equiv 0 \pmod 2 \\ \mathsf{R} < \mathsf{C} < \mathsf{L} \text{ when } \ell \equiv 1 \pmod 2). \end{array}$$

According to this order, every sequence $A \in \{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$, satisfies $\mathsf{L}\overline{\mathsf{R}} \preceq A \preceq \overline{\mathsf{R}}$; where the overline represents an infinitely repeated block.

Let $I = [-\beta, \beta]$ be an interval and $f : I \longrightarrow I$ a C^1 function. We say f is *unimodal* if there is a unique *turning point* $x_0 \in I$ such that f is decreasing in $[-\beta, x_0]$ and increasing in $[x_0, \beta]$. Clearly, x_0 is a critical point of f. We regard f as an iterative dynamical system and write $f^n(x)$ to denote the n^{th} iterate

$$\overbrace{f(f(\ldots f}^{n \text{ times}}(x)\ldots)$$

The *itinerary* of a point $x \in I$ is the infinite sequence of symbols $\iota(x) = u_1u_2\ldots \in \{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$ where

$$u_j := \begin{cases} \mathsf{L} \text{ if } f^j(x) \in [-\beta, x_0), \\ \mathsf{C} \text{ if } f^j(x_0) = x_0, \\ \mathsf{R} \text{ if } f^j(x) \in (x_0, \beta]. \end{cases}$$

Note that $\iota(f(x)) = \sigma(\iota(x))$.

It is natural to consider the order $\prec$ in this context since itineraries vary monotonically with respect to I:

Lemma 2.1. *For $x, y \in [x_0, \beta]$, $x < y$ if and only if $\iota(x) \prec \iota(y)$. Similarly, for $x, y \in [-\beta, x_0]$, $x < y$ if and only if $\iota(x) \succ \iota(y)$.*

In particular, no sequence smaller than $\iota(x_0)$ can be an itinerary for f. We call the itinerary $\iota(x_0)$ the *kneading sequence* of f and denote it by $\kappa(f)$. Obviously, κ is smaller than any of its translates: $\kappa \preceq \sigma^n(\kappa)$. Conversely, it is known that given a sequence $A \in \{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$, there exists a unimodal map g such that $\kappa(g) = A$ if and only if $A \preceq \sigma^n(A)$ for all $n \geq 1$. If this condition is satisfied, A will be called *admissible*.

Given a sequence $A \in \{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$. A_n will denote the symbol at position n.

Note that an admissible sequence that starts with R cannot contain any L. More generally, an admissible sequence A satisfies

$$\mathsf{L}\overline{\mathsf{R}} \preceq A \preceq \overline{\mathsf{L}} \preceq \overline{\mathsf{C}} \preceq \overline{\mathsf{R}}.$$

Our arguments will simplify considerably if we restrict our attention to sequences $A \prec \overline{\mathsf{L}}$. We will adopt this convention in the rest of this work.

An alternative object to encode the information carried by a kneading sequence is the *orientation kneading sequence* of f. Suppose f has kneading sequence $\kappa = u_1 u_2 \ldots$ Note that $u_n = \mathsf{L}$ if and only if $f'(f^n(x_0)) < 0$ and $u_n = \mathsf{R}$ if and only if $f'(f^n(x_0)) > 0$. By the chain rule,

$$(f^n)'(f(x_0)) = f'(f^{n-1}(x_0)) \cdot f'(f^{n-2}(x_0)) \cdot \ldots \cdot f'(f(x_0)),$$

so the sign of $(f^n)'(f(x_0))$ encodes how many times did the orbit of x_0 fall on the left sub-interval $[-\beta, x_0)$ during the first n iterates. The orientation kneading sequence will be defined as a sequence $s_1 s_2 \ldots$ of symbols $\{-.\mathsf{0}, +\}$ where $s_n = \mathsf{0}$ if $f^n(x_0) = x_0$ and $s_n = \prod_{i=1}^n \operatorname{sgn} f'(f^i(x_0))$ otherwise.

Just as before, if A' is an orientation kneading sequence, A'_n will denote the symbol at position n.

It is easy to see that the orientation kneading sequence can be constructed from the kneading sequence using the rules

(1) Since we restrict to the case $u_1 = \mathsf{L}$, we have $s_1 = -$.
(2) For $n \geq 2$, if $u_n = \mathsf{C}$ then $s_n = \mathsf{0}$,
(3) If $u_n = \mathsf{R}$ then s_n is equal to the last non-$\mathsf{0}$ symbol,
(4) If $u_n = \mathsf{L}$ then s_n is the opposite of the last non-$\mathsf{0}$ symbol.

Similarly a kneading sequence $u_1 u_2 \ldots$ can be reconstructed from an orientation kneading sequence $s_1 s_2 \ldots$ as follows:

(1) Any $\mathsf{0}$ is replaced by C.
(2) If $s_n = s_{n-1}$, then $u_n = \mathsf{R}$. otherwise, $u_n = \mathsf{L}$.

We can define an order $\prec_0$ on sequences in $\{-, \mathsf{0}, +\}^*$ that is equivalent to the order $\prec$ on $\{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$. It is enough for our purposes to define the order between a sequence $A' \in \{-, \mathsf{0}, +\}^*$ and its translates $\sigma^n(A')$. Let $A' = s_1 s_2 \ldots$ be the sequence obtained from $A \prec \overline{\mathsf{L}}$ by the above rules, then $s_1 = -$ and

(1) If $s_1 = s_{n+1}$, let $s_r \neq s_{n+r}$ be the first position where A' and $\sigma^n(A')$ differ. Then $A' \prec_0 \sigma^n(A')$ if and only if $s_r < s_{n+r}$ according to the rule $- < \mathsf{0} < +$.
(2) If $s_1 = (-1)s_{n+1}$, let $s_r = s_{n+r}$ be the first position where A' and $\sigma^n(A')$ agree. Then $A' \prec_0 \sigma^n(A')$ if and only if $s_r = -$.
(3) If $s_{n+1} = \mathsf{0}$ then $A' \prec_0 \sigma^n(A')$.

The admissibility condition for orientation kneading sequences is the same as for kneading sequences: A sequence $A' \in \{-, \mathsf{0}, +\}^*$ is admissible if and only if $A' \preceq_0 \sigma^n(A')$ for all n. With this definition, a kneading sequence is admissible if and only if the corresponding orientation kneading sequence is admissible.

2.2. **Quadratic family.** As mentioned in Lemma 2.1, any kneading sequence lies between $\mathsf{L}\overline{\mathsf{R}}$ and $\overline{\mathsf{R}}$. The following result guarantees the existence of unimodal maps with arbitrary kneading sequences.

Definition 2.1. A family $\{f_c\}$ of C^1 unimodal maps varying continuously on the parameter $c \in [a, b]$ is said to be *full* if $\kappa(f_a) = \mathsf{L}\overline{\mathsf{R}}$ and $\kappa(f_b) = \overline{\mathsf{R}}$.

Lemma 2.2. *Let $\mathcal{F} = \{f_c\}$ be a full family of unimodal maps. Then, given any admissible sequence $A \in \{\mathsf{L}, \mathsf{C}, \mathsf{R}\}^*$, there is a parameter c such that $\kappa(f_c) = A$.*

Most of our discussion is independent of the actual unimodal maps considered since we will focus only on properties derived from the associated kneading sequence. However, it is convenient to have a specific model in mind. The quadratic family $\mathcal{F} = \{f_c : x \mapsto x^2 + c\}$, where $c \in [-2, \frac{1}{4}]$, is the standard model of a full family. Indeed, for any c in the parameter range f_c has 2 fixed points $\alpha = (1 - \sqrt{1-4c})/2$, $\beta = (1 + \sqrt{1-4c})/2$ and a unique turning point at 0. The interval $I = [-\beta, \beta]$ remains bounded under iteration: $f_c(I) = I$; so f_c is unimodal.

For the map $x^2 - 2$ we have

$$0 \overset{x^-2}{\longmapsto} -2 \overset{x^2-2}{\longmapsto} 2 \overset{x^2-2}{\longmapsto} 2,$$

while for $c > 0$, $f_c^n(0) > 0$. Thus, $\kappa(f_{-2}) = \mathsf{L}\overline{\mathsf{R}}$, $\kappa(f_{\frac{1}{4}}) = \overline{\mathsf{R}}$ and $\mathcal{F}$ is a full family. A particular property of $\mathcal{F}$ that makes it a convenient family to study is monotonicity: Given $c_1, c_2 \in [-2, \frac{1}{4}]$, we have $c_1 \leq c_2$ if and only if $\kappa(f_{c_1}) \preceq \kappa(f_{c_2})$.

Perhaps more importantly, settling on the concrete family $\mathcal{F}$ to study general properties of kneading sequences we can exploit the similarities with its complex counterpart $\mathcal{F}_{\mathbb{C}} = \{z \mapsto z^2 + c\}$ where c is now an arbitrary complex parameter.

2.3. **Julia sets.** Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex polynomial map. The *filled Julia set* is defined as

$$K_f := \{z \in \mathbb{C} \mid \{f^n(z)\} \text{ remains bounded}\}.$$

The *Julia set* of f is the set $J_f := \partial K_f$. It is a compact set with a fractal structure. In the case that $f \in \mathcal{F}_{\mathbb{C}}$, the Julia set is either simply connected or totally disconnected. This dichotomy is the basis of the definition of the *Mandelbrot set*:

$$M := \{c \in \mathbb{C} \mid J_{f_c} \text{ is connected}\}.$$

The interval $[-2, 1/4]$ of real parameters for the family $\mathcal{F}$ is contained in M. In particular, for $c \in [-2, 1/4]$, the intersection of the Julia set J_c with the real axis is precisely the interval $[-\beta, \beta]$.

3. The number of periodic maps

We will be concerned with unimodal maps whose critical point is periodic, referring to them as periodic maps for short. Clearly, the corresponding kneading sequence consists of an infinitely repeating block of the form $u_1 u_2 \dots u_{n-1}\mathsf{C}$ where $u_j \in \{\mathsf{L}, \mathsf{R}\}$ and the corresponding orientation kneading sequence consists of an infinitely repeating block of the form $s_1 s_2 \dots s_{n-1} 0$ where $s_j \in \{-, +\}$. We say that n is the *exact period* of the sequence and any multiple kn will be called a *period*.

Another reason to focus on the quadratic family model is that there is a unique c for every periodic kneading sequence. Counting the total number of complex periodic maps of period n is relatively easy.

Lemma 3.1. *The number of different periodic maps in $\mathcal{F}_{\mathbb{C}}$ with period n is 2^{n-1}.*

Proof. (Gleason [DH1, exposé XIX]) Given n we want to count all the solutions c of the equation $f_c^n(0) = 0$. Since $f_c(0) = c$, this is equivalent to count the solutions of $f_c^{n-1}(c) = 0$. This is a polynomial of degree 2^{n-1} in c; hence we only need to show that all its solutions are different.

Define the family of polynomials $\{h_r \in \mathbb{Z}[z]\}$ by the recursion $h_0(z) = z$, $h_r(z) = \big(h_{r-1}(z)\big)^2 + z$, so that the critical orbit of f_c returns to 0 after n iterations if and only if $h_{n-1}(c) = 0$. Each $h_r(z)$ is a monic polynomial with integer coefficients, showing that c belongs to the ring $\mathbb{A}$ of algebraic integers.

Suppose c is a multiple root of $h_n(z)$; that is, $h_n'(c) = 0$. From $h_n'(z) = 2\big(h_{n-1}(z)\big) \cdot h_{n-1}'(z) + 1$ we conclude that $\big(h_{n-1}(c)\big) \cdot h_{n-1}'(c) = -1/2$. By the additive/multiplicative closure of $\mathbb{A}$, the left hand expression is again an algebraic integer; thus $-1/2 \in \mathbb{Q} \cap \mathbb{A} = \mathbb{Z}$ which is a contradiction (refer for instance to [D]). This contradiction shows that c must be a simple root of $h_n(z)$ and the result follows. $\square$

The corresponding result for a monotonic full family of unimodal maps is also known; see [MSS]. Our proof is based on the following definitions.

Definition 3.1. A *necklace* is a cyclic graph together with a color assignment to each vertex. In this context, vertices are called *beads*. If the necklace looks the same after rotating the beads p positions in the clockwise direction, we say that p is a *period* of the necklace. Note that n is a period for any necklace with n beads, but there can be smaller periods.

A *n-necklace* is a necklace with n beads in 2 colors, where rotation and color swap produce equivalent necklaces.

An *exact n-necklace* is an n-necklace and minimal period equal to n.

Let $p(n)$ be the number of n-necklaces and $p_0(n)$ be the number of exact n-necklaces. There are known formulas for these two functions; refer to [S]:

$$p(n) = \frac{1}{2n} \sum_{d|n} \varphi(2d) 2^{n/d}$$

$$p_0(n) = \frac{1}{2n} \sum_{\substack{d|n \\ d \equiv 1 \pmod 2}} \mu(d) 2^{n/d} \tag{1}$$

See Appendix for number theoretical definitions.

Figure 1 shows all exact 6-necklaces. The table below shows values of $p(n)$ and $p_0(n)$ for small n.

n	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	1	2	2	4	4	8	10	20	30	56	94	180
$p_0(n)$	1	1	1	2	3	5	9	16	28	51	93	170

FIGURE 1. There are 5 exact 6-necklaces. The bottom-right necklace is equivalent to necklace 5 by swapping colors and rotating 2 spaces clockwise.

Proposition 3.2. *The number of periodic admissible kneading sequences of exact period n is $p_0(n)$.*

To prove Proposition 3.2, we need the following Lemma.

Lemma 3.3. *If $K = \overline{s_1 s_2 \ldots s_{k-1} 0}$ is a periodic admissible orientation kneading sequence, then both*

$$K' = \overline{s_1 s_2 \ldots s_{k-1} +}$$

and

$$K'' = \overline{s_1 s_2 \ldots s_{k-1} -}$$

satisfy the following condition: $K' \preceq_o \sigma^n(K')$ and $K'' \preceq_o \sigma^n(K'')$ for all n.

Proof. We will prove that $K' \preceq_o \sigma^n(K')$ for all n. A similar argument applies to K''.

If K is an admissible orientation kneading sequence, then $s_1 = -$. However $\sigma^n(K)$ can begin with either $+$ or $-$. Suppose $\sigma^n(K)_1$ is $-$. Let p_n denote the first position such that $K_{p_n} \neq \sigma^n(K)_{p_n}$. Since $K \preceq_o \sigma^n(K)$, $\sigma^n(K)_{p_n} = +$ or 0 and $K_{p_n} = -$. Hence changing 0 to $+$ in K' does not affect the order.

Suppose now that $\sigma^n(K)_1 = +$. Let p_n denote the first position such that $K_{p_n} = \sigma^n(K)_{p_n}$. Since $K \preceq_o \sigma^n(K)$, $\sigma^n(K)_{p_n} = -$ or 0 and $K_{p_n} = -$. If $\sigma^n(K)_{p_n} = -$, changing 0 to $+$ does not affect the order. The case when $\sigma^n(K)_{p_n} = 0$ requires some work.

Let $K = \underbrace{s_1 s_2 \ldots s_n}_{length=n} \underbrace{s_{n+1} \ldots 0}_{length=p_n}$ and $K' = \underbrace{s_1 s_2 \ldots s_n}_{length=n} \underbrace{s_{n+1} \ldots +}_{length=p_n}$.

Since we determined $K \preceq_o \sigma^n(K)$ at position p_n, and $\sigma^n(K)_{p_n} = \mathsf{0}$, K_{p_n} must be $-$. Note also that $\sigma^n(K')_{p_n+1} = K'_1 = -$ because K' is periodic with period $p_n + n$. Therefore, K' and $\sigma^n(K')$ has the form:

$$K' = \underbrace{-......-}_{length=p_n}......$$

$$\sigma^n(K') = \underbrace{+......+}_{length=p_n}-......$$

where the corresponding signs to the left of dashed line are pairwise different.

To the right of the vertical dashed line, we have copies of $\sigma^n(K')$(above) and K'(below). Since $K \preceq_o \sigma^n(K)$, the first position with equal signs holds a $-$. When we consider the entire strings, $K' \prec_o \sigma^n(K')$. This is always true unless the first position with equal signs occurred after the position $p_n + n$, in other words, we met a $+$ in K' that is originally $\mathsf{0}$ in K before the first position with equal signs. In this case, K' and $\sigma^n k'$ has the form:

$$K' = \underbrace{-......-+...+}_{p_n+n}-...$$

$$\sigma^n(K') = \underbrace{+......+-...-}_{p_n+n}...$$

where the corresponding signs to the left of the dashed line are pairwise different.

$\sigma^n(K')_{p_n+n} = -$ because letting it be a $+$ will violate $K \preceq_o \sigma^n(K)$. To the right of the vertical dashed line, we have copies of K'(above) and $\sigma^{p_n+n}(K')$(below). Since $K \preceq_o \sigma^n(K)$, the first position with equal signs holds a $-$. When we consider the entire strings, $K' \prec_o \sigma^n(K')$. This is always true unless the first position with equal signs occurred after the position $2p_n+n$. Repeat this argument, we get $K' \prec_o \sigma^n(K')$ unless we can never find a position with equal signs for K' and $\sigma^n K'$, in which case they are equal. Therefore, $K' \preceq_o \sigma^n(K')$.

□

Proof of Proposition 3.2. Let C be the set of all real parameters c such that f_c has a periodic critical orbit of period exactly n. Let N be the set of exact n-necklaces. Note that the cardinality of C is precisely $p_0(n)$.

Let $g : C \longrightarrow N$ be the mapping such that for each $c \in C$, $g(c)$ is the necklace with n beads formed by painting $-$ as black, $+$ as white and 0 as white (the same color as $+$) clockwise.

First, we have to prove that g is well-defined; in other words, $g(c) \in N$ for all $c \in C$.

Clearly, $g(c)$ is a necklace with n beads in white and black. We only need to show that $g(c)$ has exact period n. Suppose $g(c)$ has period k and $km = n$, where m is greater than 1.

We get the same necklace when we rotate $g(c)$ with angle $\frac{2k\pi}{n}$. Therefore the orientation kneading sequence of c should be in the form:

$$p_c = (-s_2s_3..s_{k-1}+)(-s_2s_3...s_{k-1}+)...(-s_2s_3...s_{k-1}\mathsf{0}).$$

By inspection, $\sigma^{k(m-1)+1}(p_c) \prec_o p_c$. This contradicts the validity of the orientation kneading sequence, so g is indeed well defined. We will prove that g is a bijection

(A) The function g is one to one: For each necklace $g(c)$, we can read n orientation kneading sequences clockwise, depending on which position we start with. If there is a black bead, write down $-$, if there is a white bead, write down $+$. One of them is indeed the sequence p'_c modified from the orientation kneading sequence p_c by changing $\mathsf{0}$ to $+$. Since p_c is an admissible orientation kneading sequence of period exactly n, by Lemma 3.3, $p'_c \preceq_o \sigma^n(p'_c)$.

Claim 1: If c_1 and c_2 are two different periodic admissible orientation kneading sequences, then $g(c_1) \neq g(c_2)$ up to rotation.

Suppose $g(c_1)$ is a rotation of $g(c_2)$. Let the orientation kneading sequence of c_1 be p_{c_1} and the orientation kneading sequence of c_2 be p_{c_2}. The modified sequence p'_{c_1} is the shifted sequence of p'_{c_2}, because $g(c_1)$ is the rotation of $g(c_2)$.

Then for some n, $p'_{c_2} \succeq_o \sigma^n(p'_{c_1})$ and $p'_{c_2} \preceq_o \sigma^n(p'_{c_1})$. This pair of inequalities implies that either $p'_{c_2} = \sigma^n(p'_{c_1})$ or these two orientation kneading sequences are complementary in the sense that they differ at every symbol. We will denote this relation by $p'_{c_2} =_o \sigma^n(p'_{c_1})$. By Lemma 3.3, $p'_{c_1} \preceq_o \sigma^n(p'_{c_1}) =_o p'_{c_2}$. Analogously, $p'_{c_2} \preceq_o p'_{c_1}$ and hence we have $p'_{c_1} =_o p'_{c_2}$. But since both orientation kneading sequences begin with a $-$, $p'_{c_1} = p'_{c_2}$, which is a contradiction.

Claim 2: If c_1 and c_2 are two different periodic admissible orientation kneading sequences, then $g(c_1) \neq g(c_2)$ up to complement.

Suppose $g(c_1)$ is the complement of $g(c_2)$. Let p'_{c_1} and p'_{c_2} be the smallest sequences read from necklace g_{c_1} and g_{c_2}. Then p'_{c_1} and p'_{c_2} are the sequences modified from p_{c_1} and p_{c_2} by changing $\mathsf{0}$ to $+$. We know from Claim 1 that such p'_{c_1} and p'_{c_2} exist and are unique.

Without loss of generality, assume $p'_{c_2} \preceq_o p'_{c_1}$.

Let q be the complement of p'_{c_2}. We can read q from $g(c_1)$. So $q \succeq_o p'_{c_1}$ by Lemma 3.3. But $q =_o p'_{c_2} \preceq_o p'_{c_1}$. So

$$p'_{c_2} =_o q =_o p'_{c_1},$$

and

$$p_{c_1} = p_{c_2}$$

by the same reasoning as in Claim 1. Hence $c_1 = c_2$, which is a contradiction.

(B) the function g is onto: For each necklace in N, we have read off n sequences p'_i. Let q'_i be the complementary sequences of p'_i. Call the smallest sequences among $\{p'_i\}$ p'_c and let q'_c be its complementary sequence. Either p'_c or q'_c begins with a $-$.

Without loss of generality, assume p'_c begins with a $-$ and convert it back to a periodic orientation kneading sequence p_c by changing the $+$ on every nth position back to $\mathsf{0}$. The periodic orientation kneading sequence p_c is admissible because p'_c satisfies the shift condition.

Therefore, there is always some $c \in C$ that is mapped to every necklace in N. $\square$

Now we can state the main result of this Section.

Theorem 3.4. *The number of periodic admissible kneading sequences of period n is $p(n)$.*

Proof. By definition, $p(n) = \sum_{d|n} p_0(d)$. Assuming Formula 1, we have

$$p(n) = \sum_{d|n} \sum_{\substack{r|d \\ r\equiv 1 \pmod 2}} \frac{\mu(r)2^{d/r}}{2d} = \frac{1}{2n}\sum_{d|n}\frac{n}{d}\sum_{\substack{r|d \\ r\equiv 1}}\mu(r)2^{d/r} = \frac{1}{2n}\sum_{d|n} d \sum_{\substack{r|n/d \\ r\equiv 1}} \mu(r)2^{n/rd} =$$

$$\frac{1}{2n}\sum_{d|n}\sum_{\substack{r|n/d \\ r\equiv 1}} d\mu(r)2^{n/rd} = \frac{1}{2n}\sum_{D|n}\left(\sum_{\substack{rd=D \\ r\equiv 1}}\frac{D}{r}\mu(r)\right)2^{n/D} = \frac{1}{2n}\sum_{D|n}\varphi(2D)2^{n/D}.$$

using Formula 3 from the Appendix. ☐

4. Partial classification results

For the quadratic family $\mathcal{F} = \{f_c : x \mapsto x^2 + c\}$, where $c \in [-2, \frac{1}{4}]$. The interval $P = [-2, \frac{1}{4}]$ splits into 3 parts $A = [-\beta, \alpha]$, $B = [\alpha, -\alpha]$ and $C = [-\alpha, \beta]$ in such a way that both A and C map 1 to 1 onto B and B maps 2 to 1 onto A.

The interval $B \cup C$ is called the *main branch*. One can give a partial description of the behavior of a point under iteration by counting the number of times its orbit passes through the main branch.

When $f^i(0) \in C$, $f^{i+1}(0)$ is again in the main branch. However, when $f^i(0) \in B$, 2 iterates are required to return to the main branch (since $f^{i+1}(0) \in A$).

Figure 2

Definition 4.1. The *composition* of a periodic quadratic map $f \in \mathcal{F}$ with period n is a finite sequence of symbols in $\{1, 2\}$ obtained by following the orbit $0, f(0), \dots, f^{n-1}(0)$ and writing a 2 every time that $f^i(0) \in B$ and a 1 when $f^i(0) \in C$.

Note that the sum of the numerical values of the symbols in a composition is equal to n.

Remark. Intuitively, the composition describes the Left/Right position of the critical orbit relative to the fixed point α. This may seem an arbitrary choice since kneading sequences describe the Left/Right position relative to the point with lowest itinerary. The reason to introduce compositions is that they appear naturally in the context of complex quadratic maps. In the complex setting, the fixed point α splits the Julia set in several branches $Y_0, Y_1, \ldots, Y_{r-1}$. The main branch is equal to the set Y_0 that contains 0 and the β fixed point. Then, every time that the critical point lands on a "secondary" branch $f^i(0) \in Y_j$ ($j > 0$) it is forced to follow a cyclic permutation along the secondary branches and will eventually return to Y_0: $f^{i+1}(0) \in Y_{j+1}$, $f^{i+2}(0) \in Y_{j+2}, \ldots, f^{r+i-j}(0) \in Y_0$.

The definition of composition extends to the complex setting and is useful to classify different combinatorial behavior of periodic complex quadratic maps (see [AP]): Any sequence of symbols $\{\mathsf{1}, \mathsf{2}\}$ starting with $\mathsf{2}$ is the composition of some complex quadratic map. When restricting to real maps this is no longer the case, and we want to describe an attempt at classification.

Theorem 4.1. *A finite string of symbols in* $\{\mathsf{1}, \mathsf{2}\}$ *is the composition of a real unimodal map if and only if it has the following form*

1. *It contains only* $\mathsf{2}$*, or*
2. *It begins with* $\mathsf{21}$*, and no block of consecutive* $\mathsf{1}$ *is longer than the initial block of* $\mathsf{1}$*.*

Definition 4.2. The *template* of a composition is a finite string of symbols $\{{}^{\mathsf{L}}_{\mathsf{R}}, \mathsf{L}, \mathsf{R}, \mathsf{C}\}$ obtained by the following procedure:

1. If there is a $\mathsf{1}$, substitute it by R.
2. The first $\mathsf{2}$ is substituted by L.
3. Substitute any further $\mathsf{2}$ by ${}^{\mathsf{L}}_{\mathsf{R}}\mathsf{L}$.
4. Add C at the end of the sequence.

A substring of a template will be called a *sub-template.*

Lemma 4.2. *All periodic kneading sequences with given composition P are obtained by selecting either* L *or* R *on each* ${}^{\mathsf{L}}_{\mathsf{R}}$ *position of the template of P and then repeating the resulting string infinitely many times.*

Note. Not all selections give admissible kneading sequences. To find the admissible kneading sequences with the given composition, we need to plug in L or R for each ${}^{\mathsf{L}}_{\mathsf{R}}$ position in the template and test the admissibility condition for kneading sequences.

Proof. Let us describe a modified itinerary of 0 that includes the position of the point 0 itself. The first symbols are CL, accounting for the fact that $0 \in B$ and $f(0) \in A$ as specified by the initial $\mathsf{2}$ of the composition.

After the initial $\mathsf{2}$, any $\mathsf{1}$ implies that the current iterate is in C, hence, to the right of 0. Then the itinerary has an R at that position. Any $\mathsf{2}$ means that the current iterate $f^j(0)$ is in B and $f^{j+1}(0) \in A$, hence, only LL or RL can appear at that location.

Now, the true itinerary of 0 does not begin with C. Since the orbit is periodic, simply translate the symbol C to the end in order to represent $f^n(0) = 0$. Since 0 is periodic, its itinerary is periodic as claimed. $\square$

There are two basic rules when we want to fill in the $\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$ position in a template and get an admissible periodic kneading sequence.

(1) Since the template ends in C, we only need to check subtemplates shorter than n.
(2) The templates begin with LR, we only need to check subtemplates beginning with L. Those that begin with $\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}\mathsf{L}$ are automatically larger than the original template.

Proof of Theorem 4.1. We will show that if an composition satisfies neither 1) nor 2), then there is no real parameter corresponding to this composition.

Suppose the composition has some block with a consecutive $\mathsf{1}$ longer than the first block of a_1 consecutive $\mathsf{1}$. We separate the proof into 2 cases.

Case 1: $a_1 < a - 1$

Composition	2	1	...	1	2	...			
subcomposition	2	1	...	1	1	...	1	2	...

template	L	R	R	...	$\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$	L	...			
subtemplate	L	R	R	...	R	R	...	$\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$	L	...

No matter what we put in the $\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$ position, the subtemplate is always smaller. So there is no admissible kneading sequence with this given composition.

Case 2: $a_1 = a - 1$

Composition	2	1	...	2	...	
subcomposition	2	1	...	1	2	...

template	L	R	R	...	$\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$	L	...	
subtemplate	L	R	R	...	R	$\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$	L	...

If we choose L for the $\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$ position in the sub-template, the subtemplate is smaller and we can not have an admissible sequence. If we choose R for the $\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$ position in the template, we need to compare the next position.

template	L	R	R	...	R	L	...	
subtemplate	L	R	R	...	R	$\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$	L	...

If we choose R for the $\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$ position in the subtemplate, then the template is bigger and there is no admissible kneading sequence. If we choose L for this position, we need to compare the next position.

template	L	R	R	...	R	L	␣	...
subtemplate	L	R	R	...	R	L	L	...

We need to know the letter in the ␣ position in the template. If it is either C or R, then the template is bigger and there is no admissible kneading sequence. Otherwise, the corresponding number in the composition is $\mathsf{2}$ and $\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$ is in position ␣ i.e.:

template	L	R	R	...	R	L	$\genfrac{}{}{0pt}{}{\mathsf{L}}{\mathsf{R}}$	...
subtemplate	L	R	R	...	R	L	L	...

This situation was already considered. This process has to end at some point, so we can never find an admissible kneading sequence for this type of composition.

Next, we show that if a composition does satisfy 1) or 2), we have at least one real parameter corresponding to it, i.e. we have at least one valid kneading sequence that can be obtained from the composition by the method above. If the number of first block of $\mathsf{1}$ is bigger than or equal to all other blocks of $\mathsf{1}$, we can put an R at the first $\substack{\mathsf{L}\\\mathsf{R}}$ position and L at all other $\substack{\mathsf{L}\\\mathsf{R}}$ positions, then direct checking shows that this kneading sequence is admissible. $\square$

We have not been able to produce a formula to count the number of admissible kneading sequences to each composition with at least one admissible kneading sequence. But we have the following partial results.

Proposition 4.3. *Let P be a composition. If the first block of $\mathsf{1}$ is strictly longer than all other blocks of $\mathsf{1}$, any kneading sequence formed from the template of P is admissible. In other words, the number of admissible kneading sequences with the given composition is $2^{\omega-1}$, where ω is the number of $\mathsf{2}$ in P.*

Proposition 4.3 is a special case of Theorem 4.4. The proof of Proposition 4.3 will illustrate the ideas used in Theorem 4.4.

Proof. Let a_1 be the number of consecutive $\mathsf{1}$ in the first block of $\mathsf{1}$, and consider a block of $\mathsf{1}$ of length $a < a_1$. We need to analyze 2 cases.

Case 1: $a < a_1 - 1$

$$\begin{array}{lcccccccccc}
\text{Composition} & \mathsf{2} & \mathsf{1} & \dots & \mathsf{1} & \mathsf{1} & \dots & \mathsf{1} & \mathsf{2} & \dots \\
\text{subcomposition} & \mathsf{2} & \mathsf{1} & \dots & \mathsf{1} & \mathsf{2} & \dots & & &
\end{array}$$

$$\begin{array}{lcccccccccc}
\text{template} & \mathsf{L} & \mathsf{R} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{R} & \dots & \substack{\mathsf{L}\\\mathsf{R}} & \mathsf{L} & \dots \\
\text{subtemplate} & \mathsf{L} & \mathsf{R} & \mathsf{R} & \dots & \substack{\mathsf{L}\\\mathsf{R}} & \mathsf{L} & \dots & & &
\end{array}$$

No matter what we put in the $\substack{\mathsf{L}\\\mathsf{R}}$ position, the subtemplate is always bigger. So there is no obstruction to admissibility.

Case 2: $a = a_1 - 1$

$$\begin{array}{lcccccc}
\text{Composition} & \mathsf{2} & \mathsf{1} & \dots & \mathsf{1} & \mathsf{2} & \dots \\
\text{subcomposition} & \mathsf{2} & \mathsf{1} & \dots & \mathsf{2} & \dots &
\end{array}$$

$$\begin{array}{lcccccccc}
\text{template} & \mathsf{L} & \mathsf{R} & \mathsf{R} & \dots & \mathsf{R} & \substack{\mathsf{L}\\\mathsf{R}} & \mathsf{L} & \dots \\
\text{subtemplate} & \mathsf{L} & \mathsf{R} & \mathsf{R} & \dots & \substack{\mathsf{L}\\\mathsf{R}} & \mathsf{L} & \dots &
\end{array}$$

If we choose L for the $\substack{\mathsf{L}\\\mathsf{R}}$ position in the sub-template, the subtemplate is bigger and there is no obstruction to admissibility. If we choose R for the $\substack{\mathsf{L}\\\mathsf{R}}$ position in the sub-template, we need to compare the next position.

$$\begin{array}{lcccccccc}
\text{template} & \mathsf{L} & \mathsf{R} & \mathsf{R} & \dots & \mathsf{R} & \substack{\mathsf{L}\\\mathsf{R}} & \mathsf{L} & \dots \\
\text{subtemplate} & \mathsf{L} & \mathsf{R} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \dots &
\end{array}$$

If we choose R for the $\substack{\mathsf{L}\\\mathsf{R}}$ position in the template, then the template is smaller and there is no obstruction. If we choose L, we need to compare the next position.

$$\begin{array}{lcccccccc}
\text{template} & \mathsf{L} & \mathsf{R} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \mathsf{L} & \dots \\
\text{subtemplate} & \mathsf{L} & \mathsf{R} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \text{\textvisiblespace} & \dots
\end{array}$$

We need to determine the letter in the ␣ position in the subtemplate. If it is C or R, then the template is smaller and there is no obstruction. Otherwise, the corresponding number in the composition is 2 and position ␣ of the subtemplate holds $^{\mathsf{L}}_{\mathsf{R}}$:

template	L	R	R	...	R	L	L	...
subtemplate	L	R	R	...	R	L	$^{\mathsf{L}}_{\mathsf{R}}$	...

This case was already considered. Since no substitution at $^{\mathsf{L}}_{\mathsf{R}}$ positions yields an obstruction, the kneading sequence is admissible. □

Definition 4.3. Consider composition $h = k_1k_2\dots k_n$, where $k_i = \{1, 2\}$. The composition h' obtained by removing a 1 from each block of consecutive 1 in h is the *reduction* of h. Suppose h contains a subcomposition $k_r \dots k_{r+s}$ such that $k_{r+j} = k_{j+1}$ for $j = 0, 1, \dots, s$, but $k_{r+s+1} \neq k_{s+1}$. If k_{r+s+1} or k_{s+1} is an isolated 1 (i.e. surrounded by 2) we say that h is *blocked.*

Theorem 4.4. *If a composition h is not blocked, then there is a bijection between the set of kneading sequences of h and the set of kneading sequences of h' (the reduction of h).*

To prove the theorem 4.4 we need the following Definition and Lemma.

Definition 4.4. Let $c = s_1s_2\dots s_n$ be a composition and $\overline{c} = s_r \dots s_{r+m}$ be a subcomposition such that $s_{r+j} = s_{j+1}$ for $j = 0, 1, \dots, m$, but $s_{r+m+1} \neq s_{m+1}$. Let k and $\overline{k}$ be a kneading sequence template and subtemplate corresponding to c and $\overline{c}$, respectively. In what follows, $\overline{c}_j$ will denote the j^{th} symbol of $\overline{c}$, that is $\overline{c}_j = s_{r+j-1}$.

We say that $k \prec \overline{k}$ up to the $(m + 1)^{\text{st}}$ position if one of the following conditions applies.

(1) $k_n < \overline{k}_n$, where $n < m$ and n is the first position where k and $\overline{k}$ are different.
(2) $k_n = \overline{k}_n$ for all $n \leq m$, the number of L up to position m is odd, $c_{m+1} = 1$ and $\overline{c}_{m+1} = 2$.
(3) $k_n = \overline{k}_n$ for all $n \leq m$, the number of L up to position m is even, $c_{m+1} = 2$ and $\overline{c}_{m+1} = 1$.

Lemma 4.5. *Let $c = s_1s_2\dots s_n$ be a composition and $\overline{c} = s_r \dots s_{r+m}$ be a subcomposition such that $s_{r+j} = s_{j+1}$ for $j = 0, 1, \dots, m$, but $s_{r+m+1} \neq s_{m+1}$. Let k and $\overline{k}$ be a kneading sequence template and subtemplate corresponding to c and $\overline{c}$, respectively. Then $k \prec \overline{k}$ if and only if $k \prec \overline{k}$ up to the $(m + 1)^{\text{st}}$ position.*

Proof. If $k_n \neq \overline{k}_n$ for some $n \in \{1, \dots, m\}$, obviously $k \prec \overline{k}$ if and only if $k \prec \overline{k}$ up to the $(m + 1)^{\text{st}}$ position. From now on, we assume that $k_n = \overline{k}_n$ for all $n = 1, \dots, m$. We separate the proof into 2 cases.

Case 1: <u>The number of L up to position m is odd, $c_{m+1} = 1$ and $\overline{c}_{m+1} = 2$.</u>

Composition	2	1	...	1	...
Subcomposition	2	1	...	2	...

Template	L	R	...	R	...	
Subtemplate	L	R	...	$^{\mathsf{L}}_{\mathsf{R}}$	L	...

If $^{\mathsf{L}}_{\mathsf{R}} = \mathsf{L}$, $k \prec \overline{k}$ since the number of L before this position is odd.
If $^{\mathsf{L}}_{\mathsf{R}} = \mathsf{R}$, we need to compare the next position.

$$\begin{array}{cccccc} \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \textvisiblespace & \dots \\ \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \dots \end{array}$$

We need to determine the letter in the $\textvisiblespace$ position. If it is C or R we are done: $k \prec \overline{k}$. Otherwise the corresponding number in the composition is 2 so the template holds a $\substack{\mathsf{L}\\\mathsf{R}}$ in the $\textvisiblespace$ position. Thus the picture is

$$\begin{array}{ccccccc} \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \substack{\mathsf{L}\\\mathsf{R}} & \mathsf{L} & \dots \\ \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \dots & \end{array}$$

If $\substack{\mathsf{L}\\\mathsf{R}}$ is R, $k \prec \overline{k}$ because the number of L before this position is odd. Otherwise we need to compare the next position.

$$\begin{array}{ccccccc} \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \mathsf{L} & \dots \\ \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \textvisiblespace & \dots \end{array}$$

If the $\textvisiblespace$ holds a C or R, we are done because the number of L before this position is even. Otherwise the corresponding number in the composition is 2 so the template holds a $\substack{\mathsf{L}\\\mathsf{R}}$ in the $\textvisiblespace$ position. Thus the picture is

$$\begin{array}{ccccccc} \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \mathsf{L} & \dots \\ \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \substack{\mathsf{L}\\\mathsf{R}} & \dots \end{array}$$

and the previous argument applies. Finally this procedure stops at some point because the sequence is periodic.

Case 2: <u>The number of L up to position m is even; $c_{m+1} = 2$ and $\overline{c}_{m+1} = 1$.</u>

The proof of this case is essentially the same as in Case 1. But this time, we begin matching a $\substack{\mathsf{L}\\\mathsf{R}}$ located in the template:

$$\begin{array}{llllll} \text{Composition} & 2 & 1 & \dots & 2 & \dots \\ \text{Subcomposition} & 2 & 1 & \dots & 1 & \dots \end{array}$$

$$\begin{array}{lllllll} \text{Template} & \mathsf{L} & \mathsf{R} & \dots & \substack{\mathsf{L}\\\mathsf{R}} & \mathsf{L} & \dots \\ \text{Subtemplate} & \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \dots & \end{array}$$

If $\substack{\mathsf{L}\\\mathsf{R}} = \mathsf{L}$, $k \prec \overline{k}$ since the number of L before this position is even.
If $\substack{\mathsf{L}\\\mathsf{R}} = \mathsf{R}$, we need to compare the next position.

$$\begin{array}{cccccc} \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \dots \\ \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \textvisiblespace & \dots \end{array}$$

We need to determine the letter in the $\textvisiblespace$ position. If it is C or R we are done: $k \prec \overline{k}$. Otherwise the corresponding number in the composition is 2 so the subtemplate holds a $\substack{\mathsf{L}\\\mathsf{R}}$ in the $\textvisiblespace$ position. Thus the picture is

$$\begin{array}{ccccccc} \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \dots & \\ \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \substack{\mathsf{L}\\\mathsf{R}} & \mathsf{L} & \dots \end{array}$$

If $\substack{\mathsf{L}\\\mathsf{R}}$ is R, $k \prec \overline{k}$ because the number of L before this position is even. Otherwise we need to compare the next position.

$$\begin{array}{ccccccc} \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \textvisiblespace & \dots \\ \mathsf{L} & \mathsf{R} & \dots & \mathsf{R} & \mathsf{L} & \mathsf{L} & \dots \end{array}$$

If the $\llcorner\!\lrcorner$ holds a C or R, we are done because the number of L before this position is even. Otherwise the corresponding number in the composition is 2 so the template holds a $\substack{\mathsf{L}\\\mathsf{R}}$ in the $\llcorner\!\lrcorner$ position. Thus the picture is

$$\begin{array}{cccccc} \mathsf{L} & \mathsf{R} & \ldots & \mathsf{R} & \mathsf{L} & \substack{\mathsf{L}\\\mathsf{R}} & \ldots \\ \mathsf{L} & \mathsf{R} & \ldots & \mathsf{R} & \mathsf{L} & \mathsf{L} & \ldots \end{array}$$

Therefore $k \prec \overline{k}$ if $k \prec \overline{k}$ up to the $(m+1)^{\text{st}}$ position.

Conversely, we need to show that if $k \prec \overline{k}$, then $k \prec \overline{k}$ up to the $(m+1)^{\text{st}}$ position. Suppose k is NOT smaller than $\overline{k}$ up to the $(m+1)^{\text{st}}$ position. This can happen in 2 different ways:

The number of L up to position m is even, $c_{m+1} = 1$ and $\overline{c}_{m+1} = 2$.

The number of L up to position m is odd, $c_{m+1} = 2$ and $\overline{c}_{m+1} = 1$.

Since the parity of the number of L is opposite to that in Cases 1 and 2, following the argument for those cases yields that $k \succ \overline{k}$; this is a contradiction. Thus, $k \prec \overline{k}$ implies that $k \prec \overline{k}$ up to the $(m+1)^{\text{st}}$ position.

□

Proof of Theorem 4.4. A 1 in the composition is converted to R in the kneading sequence template. The order between a kneading sequence and its translate depends on the number of L before the position where they differ. Because the composition is not blocked, when we remove a 1 from each block of 1, the number of L in front of the differing position remains the same. This will also be the case after reducing the template and the corresponding subtemplate. Now we have to show that the symbols at those locations where k and $\overline{k}$ first differ, are the same before and after the reduction.

By Lemma 4.5, we know that $k \prec \overline{k}$ if and only if $k \prec \overline{k}$ up to the $(m+1)^{\text{st}}$ position; where $m+1$ is the position where c and $\overline{c}$ first differ.

It follows that k and $\overline{k}$ are reduced by the same number of R. Therefore a kneading sequence for c is valid if and only if the reduced kneading sequence is valid for the reduced c. That is, there is a 1 to 1 correspondence between the kneading sequence of c and the kneading sequence of c'. □

Appendix

Definition 4.5. The greatest common divisor between 2 integers m and n is denoted (m, n). The Möbius function $\mu : \mathbb{N} \longrightarrow \mathbb{N}$ is determined by $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ for $n \geq 2$. Alternatively, it can be defined by $\mu(n) = 0$ if n contains a repeated factor and $\mu(n) = (-1)^r$ when n is the product of r different primes. The Euler *totient* function $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ counts, for an arbitrary n, the number of integers $m \leq n$ such that $(m, n) = 1$.

The Möbius function plays a fundamental role in number theory due to the following property, known as the Möbius inversion formula.

Proposition 4.6. *Consider a function* $f : \mathbb{N} \longrightarrow \mathbb{C}$ *and define* $g(n) = \sum_{d|n} f(d)$. *Then*

$$f(n) = \sum_{d|n} g(n/d)\mu(d).$$

As an application, partition the set $S = \{1, 2, \ldots, n\}$ into the subsets $S_d = \{m \leq n \mid (m, n) = d\}$ for every divisor d of n. The S_d are disjoint and their union is S. Moreover, the size of S_d is $\varphi(n/d)$. Thus,

$$n = \sum_{d|n} \sum_{\substack{m \leq n \\ (m,n)=d}} 1 = \sum_{d|n} \varphi(d)$$

and the Möbius inversion formula shows that

$$\varphi(n) = \sum_{d|n} (n/d)\mu(d). \tag{2}$$

Our goal in this Appendix is to derive a formula for $\varphi(2n)$ by splitting (2) in 2 sums for even and odd divisors. Note that an odd divisor of $2n$ is a (odd) divisor of n, whereas an even divisor of $2n$ either has a 2^2 factor and the corresponding μ term vanishes, or it has a single factor of 2, in which case it is of the form $2d$ where d is an odd divisor of n.

$$\varphi(2n) = \sum_{d|2n} \frac{2n}{d}\mu(d) = \sum_{\substack{d|2n \\ d\equiv 1}} \frac{2n}{d}\mu(d) + \sum_{\substack{d|2n \\ d\equiv 0}} \frac{2n}{d}\mu(d) = \sum_{\substack{d|n \\ d\equiv 1}} \frac{2n}{d}\mu(d) + \sum_{\substack{d|n \\ d\equiv 1}} \frac{2n}{2d}\mu(2d)$$

Since $\mu(2d) = -\mu(d)$ for odd d, we obtain

$$\varphi(2n) = \sum_{\substack{d|n \\ d\equiv 1}} \frac{2n}{d}\mu(d) - \frac{n}{d}\mu(d) = \sum_{\substack{d|n \\ d\equiv 1}} \frac{n}{d}\mu(d). \tag{3}$$

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